LIESMARS

Course Title

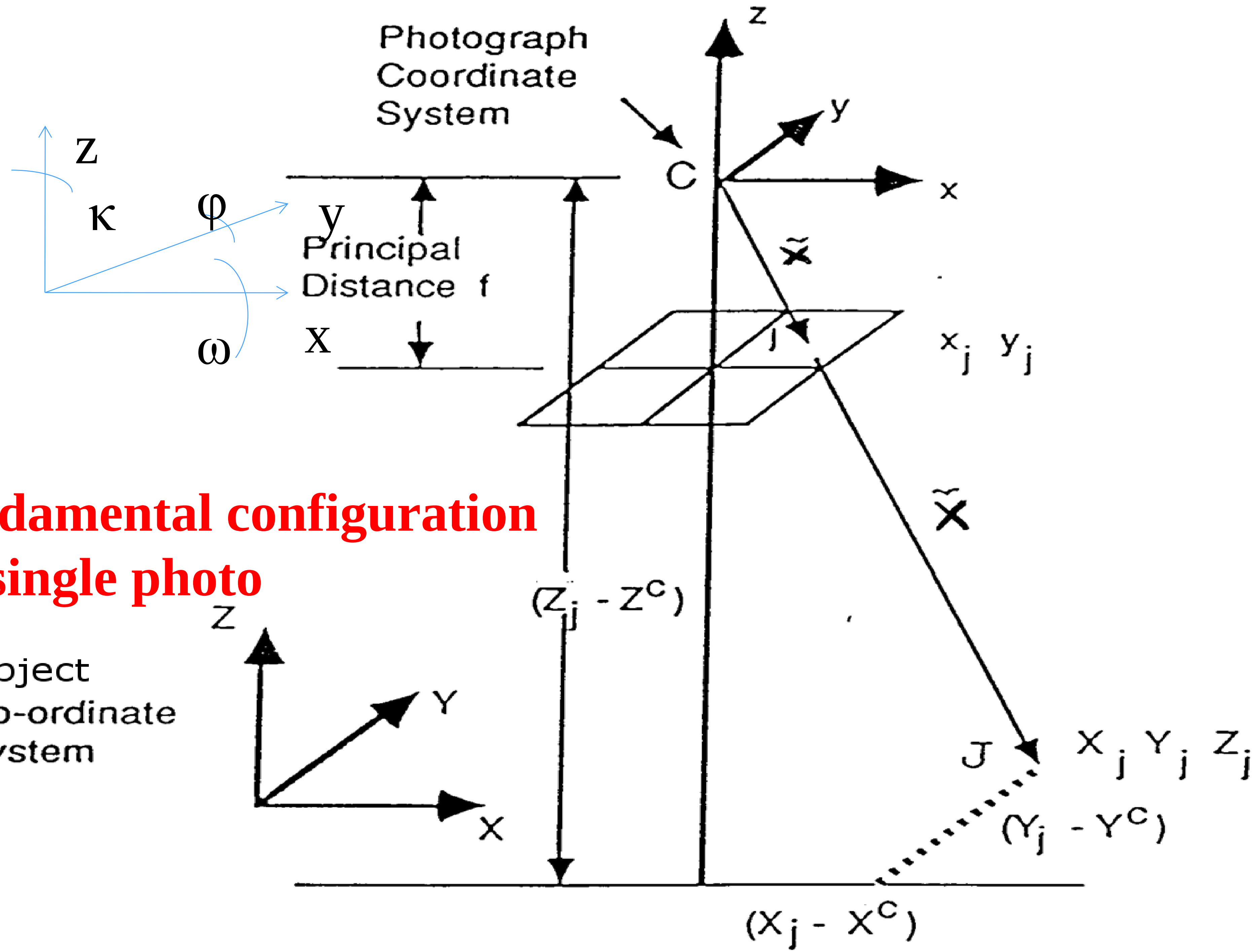
Principles of Photogrammetry

Chapter 3.1

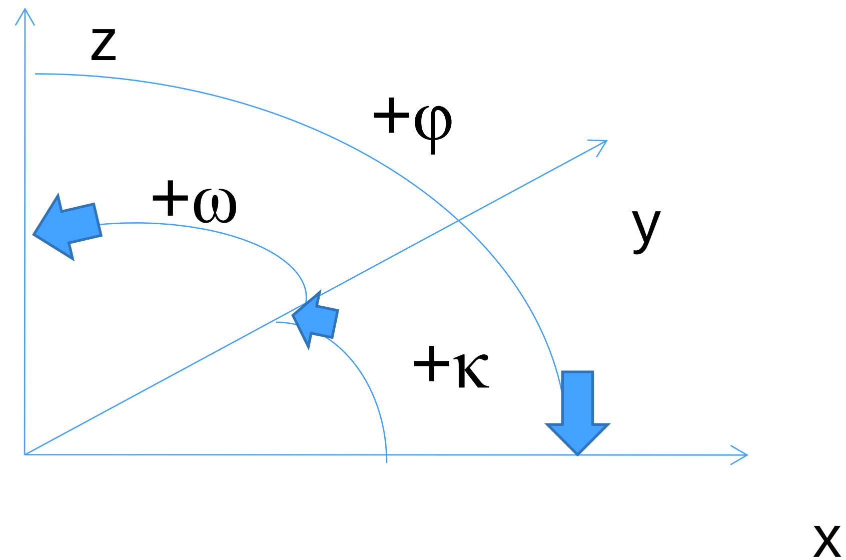
Mathematics of Analytical Photogrammetry & Block Adjustment

Introduction

- Analytical photogrammetry involves computations for the determination of 3D object coordinates from image coordinates on overlapping photographs.
- The fundamental formulae are known as the collinearity equations.
- Includes the determination of the exterior orientation of the photos.
- Description separate for frame aerial and pushbroom images.
- Push-broom images require a modified approach



Fundamental configuration for single photo



Positive directions of camera rotations

Definition of vectors in image and object sp

Image Space vector components

$$\mathbf{x}_j = \begin{bmatrix} x_j & - & x_0 \\ y_j & - & y_0 \\ & - & f \end{bmatrix}$$

Ground or Object space vector components

$$X_j = \begin{bmatrix} X_j - X^c \\ Y_j - Y^c \\ Z_j - Z^c \end{bmatrix}$$

Where X^c, Y^c, Z^c are the coordinates
of the perspective centre of the camera
and X_j, Y_j and Z_j are the coordinates
of j in the object space.

Relationship between image and object coordinate systems

The differences in orientations of the image and object coordinate systems are expressed by the tilts ω, φ and κ about the image x, y and z axes respectively.

These are the rotations required to rotate an untilted coordinate system referred to as **x***, centred on the perspective centre, into the tilted system **x**

That is, after introduction of the 3 rotations, **x*** will be rotated so that it is parallel to the actual titled coordinate photograph system , **x**.

The effects of the rotations of ω, φ, and κ can be individually expressed by orthogonal rotation matrices, **M**, of the form:-

$$\mathbf{M}_{\omega} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix} \tag{3.3}$$

$$\mathbf{M}_{\phi} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \tag{3.4}$$

$$\mathbf{M}_{\kappa} = \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.5}$$

Use the convention that the relationship between the two coordinate systems **x** and **x***, and **M** is expressed in the following order:-

$$\mathbf{x} \quad = \quad \mathbf{M}\,\mathbf{x}^*$$

where **M** refers to any one of the above rotation matrices

A combination of all 3 rotations can be determined by successively multiplying the matrices together.

Adopt the order of occurrence of the tilts as $\omega \rightarrow \varphi \rightarrow \kappa$

Important to be consistent with this order

Apply ω rotation about the x* axis.

$$\mathbf{x}_{\omega} = \mathbf{M}_{\omega}\,\mathbf{x^*} \qquad 3.6$$

Apply φ rotation on the $\mathbf{x}_{\boldsymbol{\omega}}$ vector about $y_{\boldsymbol{\omega}}$ - axis, that is the new position of the y coordinate after the ω rotation

$$\begin{aligned}\mathbf{x}_{\varphi} &= \mathbf{M}_{\varphi}\,\mathbf{x}_{\omega} \\ &= \mathbf{M}_{\varphi}\,\mathbf{M}_{\omega}\,\mathbf{x^*} \qquad 3.7\end{aligned}$$

Apply κ rotation on the $\mathbf{x}_{\omega\varphi}$ vectors about $z_{\omega\varphi}$ - axis, that is the new position of the z coordinate after the ω and φ rotations

$$\mathbf{x}_{\omega\varphi\kappa} = \mathbf{M}_{\kappa}\, \mathbf{x}_{\omega\varphi}$$

Hence x = Mκ Mφ Mω x* 3.8

$$\text{Combined Matrix M} = \begin{bmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} & \mathbf{m}_{13} \\ \mathbf{m}_{21} & \mathbf{m}_{22} & \mathbf{m}_{23} \\ \mathbf{m}_{31} & \mathbf{m}_{32} & \mathbf{m}_{33} \end{bmatrix}$$

$$
\begin{aligned}
m_{11} &= \cos\phi \cos\kappa \\
m_{12} &= \cos\omega \sin\kappa + \sin\omega \sin\phi \cos\kappa \\
m_{13} &= \sin\omega \sin\kappa - \cos\omega \sin\phi \cos\kappa \\
m_{21} &= -\cos\phi \sin\kappa \\
m_{22} &= \cos\omega \cos\kappa - \sin\omega \sin\phi \sin\kappa \\
m_{23} &= \sin\omega \cos\kappa + \cos\omega \sin\phi \sin\kappa \\
m_{31} &= \sin\phi \\
m_{32} &= -\sin\omega \cos\phi \\
m_{33} &= \cos\omega \cos\phi
\end{aligned}
$$

The **M** matrix is orthogonal which means that its inverse equals its transpose

i.e. $\mathbf{M}^{-1} = \mathbf{M}^{T}$

Referring again to equations 3.1 and 3.2, the relationship between these coordinates can be written as:

$$\mathbf{x}_j = \lambda \, \mathbf{M} \, \mathbf{X}_j \qquad 3.9$$

where λ is a scale factor between the two sets of coordinates

This relationship can be written in full

$$\begin{bmatrix} x_j - x_o \\ y_j - y_o \\ - f \end{bmatrix} = \lambda_j \; \mathbf{M} \begin{bmatrix} X_j - X^c \\ Y_j - Y^c \\ Z_j - Z^c \end{bmatrix}$$

Let us also add additional parameters Δx_j, Δy_j to the image coordinates which represent errors (either known or unknown) in the image coordinates x_j and y_j

$$\begin{bmatrix} x_j - x_0 + \Delta x_j \\ y_j - y_0 + \Delta y_j \\ -f \end{bmatrix} = \lambda_j \, \mathbf{M} \begin{bmatrix} X_j - X^c \\ Y_j - Y^c \\ Z_j - Z^c \end{bmatrix}$$

Dividing each of the first 2 equations by the 3rd

Collinearity Equations

$$x_j - x_0 + \Delta x_j \quad = \quad \frac{- f\,[\, m_{11}\,(X_j - X^c) + m_{12}\,(Y_j - Y^c) + m_{13}\,(Z_j - Z^c)]}{m_{31}\,(X_j - X^c) + m_{32}\,(Y_j - Y^c) + m_{33}\,(Z_j - Z^c)}$$

$$y_j - y_0 + \Delta y_j \quad = \quad \frac{- f\,[\, m_{21}\,(X_j - X^c) + m_{22}\,(Y_j - Y^c) + m_{23}\,(Z_j - Z^c)]}{m_{31}\,(X_j - X^c) + m_{32}\,(Y_j - Y^c) + m_{33}\,(Z_j - Z^c)}$$

x_j, y_j are image coordinates of object j

x_0 , y_0 are displacement coordinates between the actual origin of the image coordinates and the true origin

Δx_j , Δy_j are the corrections applied to the image coordinates

f is the camera principal distance.

X_j , Y_j , Z_j are the object coordinates of point j

X^c , Y^c , Z^c , are the coordinates of the camera in the object space coordinate system

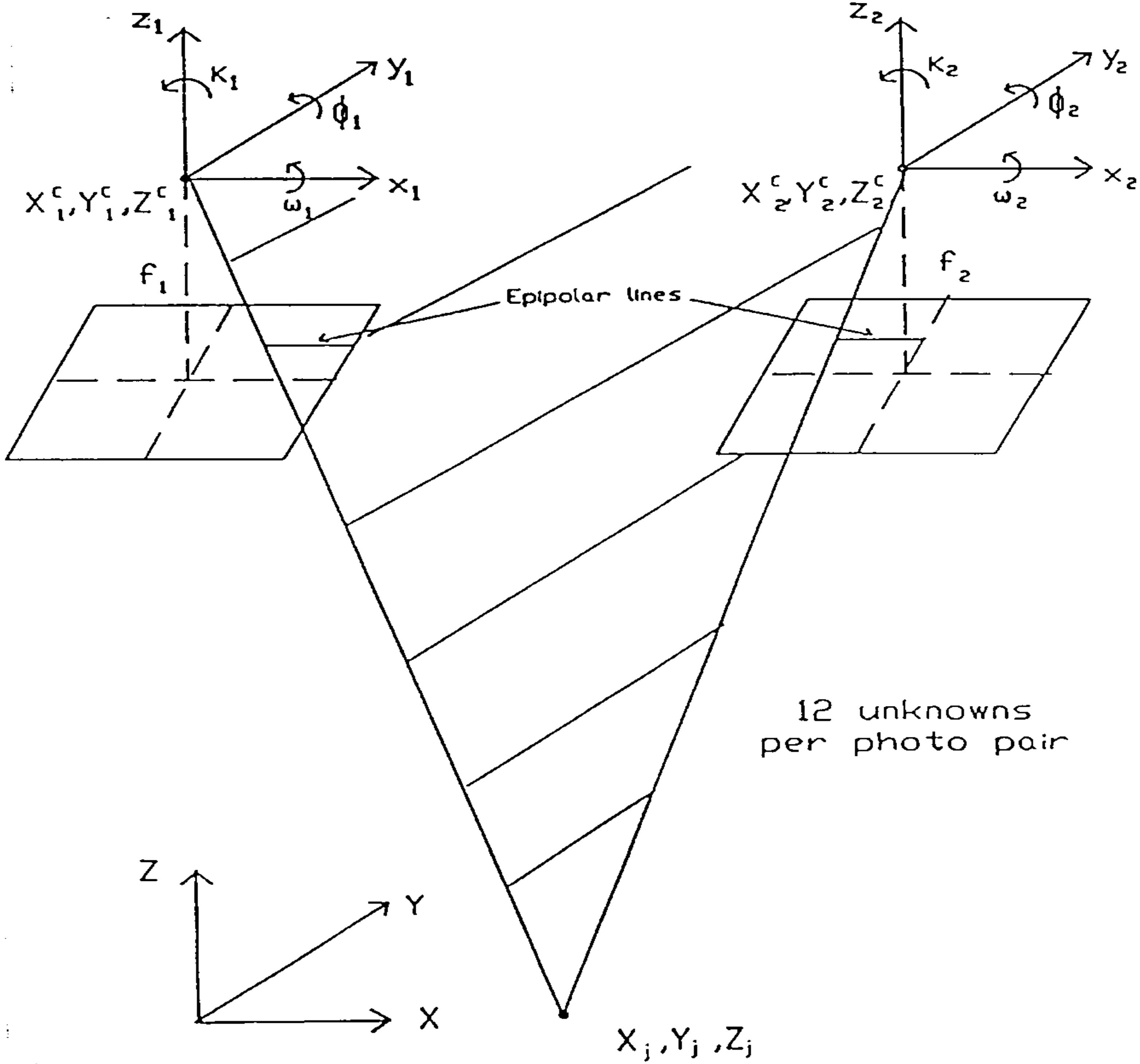
m_{11} ... m_{33} are the elements of rotation matrix **M**

Rotation Matrix M

$$M_i = \begin{bmatrix} \cos\phi \cos\kappa & \cos\omega \sin\kappa + \sin\omega \sin\phi \cos\kappa & \sin\omega \sin\kappa - \cos\omega \sin\phi \cos\kappa \\ -\cos\phi \sin\kappa & \cos\omega \cos\kappa - \sin\omega \sin\phi \sin\kappa & \sin\omega \cos\kappa + \cos\omega \sin\phi \sin\kappa \\ \sin\phi & -\sin\omega \cos\phi & \cos\omega \cos\phi \end{bmatrix}$$

Solution of Exterior Orientation – Space Resection – 2 photos only

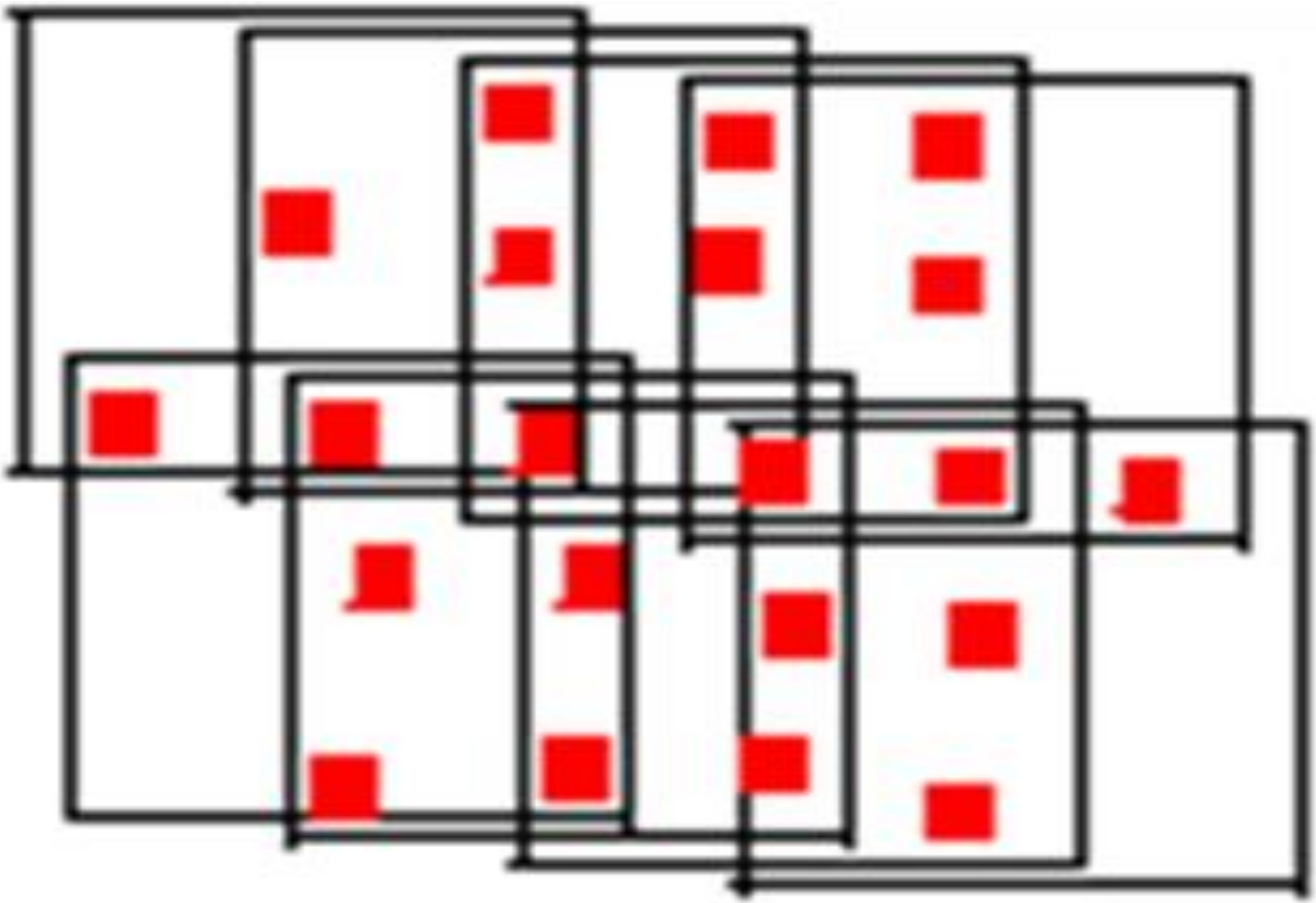
- Need to determine camera position and tilts
- GNSS is available to determine positions
- IMU/INS may be available to determine tilts
- If both are available, Direct Orientation is possible provided the data are accurate – then no control points needed
- Alternatively can use GNSS/IMU data with control points to check the results and improve accuracy
- If no GNSS/IMU available, need a minimum of 3 control points, preferably 4



Z_1
K_1
Y_1
ϕ_1
X_1
ω_1
X^c_1, Y^c_1, Z^c_1
f_1
Z_2
K_2
Y_2
ϕ_2
X_2
ω_2
X^c_2, Y^c_2, Z^c_2
f_2
Epipolar lines
12 unknowns
per photo pair
Z
Y
X
X_j, Y_j, Z_j

Aerial Triangulation – Control Extension

- Use geometry of the photographs together with a small number of ground control points
- Significantly reduces the number of ground control points
- Determine the locations of a dense set of points on all photographs
- Carried out on so-called 'blocks' of photographs, comprising a number of strips of photographs
- Minimum size block is 2 photos
- No limit to the maximum size of blocks



Aerial Triangulation

- **Points selected and measured on the photographs**
  - Control points (Ground or Object)
    - Targets
    - Natural features
  - Tie points
    - Multiple prominent points but not necessarily identifiable features
    - Selected by software and referred to image coordinate system
    - Tie points are also required to connect the strips together
    - Special procedures used to extract and match the points - current methods are based on such approaches as ‘Structure from Motion’ SfM.
- **Block adjustment**
  - Image coordinates measured automatically by software and input to block adjustment

Adjustment of Multiple Overlapping Photographs - Block Adjustment

- In most computations in geomatics, it is typical to take more observations than that the minimum necessary for a solution
- Reason:
  - Ensure that blunders are detected
  - Improve the accuracy of the final result
  - Include observations of varying accuracies
  - Speed up the solution by incorporating blunder detection methods
- A least squares solution is typically used to manage redundant observations, and determine accuracies of the solution

Adjustment of Aerial Triangulation – Block Adjustment

- A least squares solution minimizes the weighted sum of squares of the residuals
- Weights are a function of the accuracies of the observations described by the variance = (standard deviation)^2
- Least squares solution can only be applied on linear problems
- Method of least squares involves formulation of ‘normal equations’
- Solution determines
  - Corrections to all image coordinate observations
  - Estimates of the exterior orientation of all photos and all object point coordinates determined

Linearization of collinearity equations

- The collinearity equations are highly non-linear
- Must be linearized by Taylor's Series by partial differentiation
- Iterative solution
- Start with some approximate values of the unknown and determine corrections ΔX to those values
- Note that in ΔX will be written as $\hat{\Delta X}$ indicating that it is an estimate only

Procedure for solution of block adjustment

- The least squares mathematical model they will be expressed in the form of:

$$A.\Delta X - b = v$$

where

- A is referred to the Jacobian, coefficient or design, matrix derived by linearizing collinearity equations
- ΔX is the vector of corrections applied to the assumed approximate values of the unknowns.
- b equals ℓ - AX i.e. the vector of observations minus the evaluation of the equation based on the approximate values of the parameters.
- v is the vector of residuals to the observations.

Basic formulae for observations

- The "true" image coordinates x and y are written in terms of measured quantities x°°, y°° and residuals v_x and v_y also including subscript ‘j’ for object points and ‘i’ for photograph number.

$$x_{ij} = x_{ij}^{\circ\circ} + v_{xij}$$

$$y_{ij} = y_{ij}^{\circ\circ} + v_{yij}$$

$$\text{i.e} \quad L = \ell + v$$

Basic formulae for parameters

- The unknown parameters are expressed in terms of a correction to an approximate value

$$\hat{X} = X^o + \Delta \hat{X}$$

- For example for

  Camera parameters

$$\begin{bmatrix} \hat{X}_i^c \\ \hat{Y}_i^c \\ \hat{Z}_i^c \\ \hat{\omega}_i \\ \hat{\phi}_i \\ \hat{\kappa}_i \end{bmatrix} = \begin{bmatrix} X^c \\ Y^c \\ Z^c \\ \omega^c \\ \varphi^c \\ \kappa^c \end{bmatrix}_i^o + \begin{bmatrix} \Delta \hat{X}_i^c \\ \Delta \hat{Y}_i^c \\ \Delta \hat{Z}_i^c \\ \Delta \hat{\omega}_i \\ \Delta \hat{\phi}_i \\ \Delta \hat{\kappa}_i \end{bmatrix}$$

For object points

$$\begin{bmatrix} \hat{X}_j \\ \hat{Y}_j \\ \hat{Z}_j \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_j^o + \begin{bmatrix} \Delta \hat{X}_j \\ \Delta \hat{Y}_j \\ \Delta \hat{Z}_j \end{bmatrix}$$

Collinearity equations for a single point

- Basic equation for least squares adjustment
- $A.\Delta X - b = v$ **Linearized equations**

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{pmatrix}_{ij} \begin{pmatrix} \Delta\hat{X}^c \\ \Delta\hat{Y}^c \\ \Delta\hat{Z}^c \\ \Delta\hat{\omega} \\ \Delta\hat{\phi} \\ \Delta\hat{\kappa} \end{pmatrix}_i + \begin{pmatrix} a_{17} & a_{18} & a_{19} \\ a_{27} & a_{28} & a_{29} \end{pmatrix}_{ij} \begin{pmatrix} \Delta\hat{X} \\ \Delta\hat{Y} \\ \Delta\hat{Z} \end{pmatrix}_j - \begin{pmatrix} b_x \\ b_y \end{pmatrix}_{ij} = \begin{pmatrix} v_{xij} \\ v_{yij} \end{pmatrix}$$

Matrix form

$$\underset{(2,\,6)}{\dot{A}_{ij}} \quad \underset{(6,\,1)}{\dot{\Delta X}_i} + \underset{(2,\,3)}{\ddot{A}_{ij}} \quad \underset{(3,\,1)}{\ddot{\Delta X}_j} - \underset{(2,\,1)}{b_{ij}} = \underset{(2,\,1)}{v_{ij}}$$

Adding Equations for Multiple Photos and Points

- Add m photographs

$$
\begin{bmatrix} \dot{A}_{1j} & & & \\ & \dot{A}_{2j} & & \\ & & \dot{A}_{3j} & \\ & & & \dot{A}_{mj} \end{bmatrix}
\begin{bmatrix} \dot{\Delta X}_1 \\ \dot{\Delta X}_2 \\ \dot{\Delta X}_3 \\ \bullet \\ \bullet \\ \bullet \\ \dot{\Delta X}_m \end{bmatrix}
+
\begin{bmatrix} \ddot{A}_{1j} \\ \ddot{A}_{2j} \\ \ddot{A}_{3j} \\ \bullet \\ \bullet \\ \bullet \\ \ddot{A}_{mj} \end{bmatrix}
\ddot{\Delta X}_j -
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \bullet \\ \bullet \\ \bullet \\ b_m \end{bmatrix}
=
\begin{bmatrix} v_{1j} \\ v_{2j} \\ v_{3j} \\ \bullet \\ \bullet \\ \bullet \\ v_{mj} \end{bmatrix}
$$

$$
\dot{A}_j \quad \dot{\Delta X} \quad + \quad \ddot{A}_j \quad \ddot{\Delta X}_j \quad - \quad b_j = \quad v_j
$$

(2m,6m) (6m,1) (2m,3) (3,1) (2m,1) (2m,1)

Adding Equations for Multiple Photos and Points

- **Extend equations to n points on the m photographs**

$$\begin{bmatrix}\dot{A}_1\\ \dot{A}_2\\ \dot{A}_3\\ \bullet\\ \bullet\\ \bullet\\ \bullet\\ \dot{A}_n\end{bmatrix}\Delta\dot{X} + \begin{bmatrix}\ddot{A}_1 & & & \\ & \ddot{A}_2 & & \\ & & \ddot{A}_3 & \\ & & & \ddot{A}_n\end{bmatrix}\begin{bmatrix}\Delta\ddot{X}_1\\ \Delta\ddot{X}_2\\ \Delta\ddot{X}_3\\ \bullet\\ \bullet\\ \bullet\\ \Delta\ddot{X}_n\end{bmatrix} - \begin{bmatrix}b_1\\ b_2\\ b_3\\ \bullet\\ \bullet\\ \bullet\\ \bullet\\ b_n\end{bmatrix} = \begin{bmatrix}v_1\\ v_2\\ v_3\\ \bullet\\ \bullet\\ \bullet\\ \bullet\\ v_n\end{bmatrix}$$

$$\dot{A}\quad \Delta\dot{X} \quad + \quad \ddot{A} \quad \Delta\ddot{X} \quad - \quad b \quad = \quad v$$

(2mn, 6m) (6m, 1) (2mn, 3n) (3n, 1) (2mn, 1) (2mn, 1)

Adding Equations for Multiple Photos and Points

- **Extend equations to n points on the m photographs**

$$\begin{bmatrix} \dot{A}_1 \\ \dot{A}_2 \\ \dot{A}_3 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ \dot{A}_n \end{bmatrix} \Delta\dot{X} + \begin{bmatrix} \ddot{A}_1 & & & \\ & \ddot{A}_2 & & \\ & & \ddot{A}_3 & \\ & & & \\ & & & \ddot{A}_n \end{bmatrix} \begin{bmatrix} \Delta\ddot{X}_1 \\ \Delta\ddot{X}_2 \\ \Delta\ddot{X}_3 \\ \bullet \\ \bullet \\ \bullet \\ \Delta\ddot{X}_n \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ b_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \bullet \\ \bullet \\ \bullet \\ \bullet \\ v_n \end{bmatrix}$$

$$\underset{(2mn,\ 6m)}{\dot{A}} \ \underset{(6m,\ 1)}{\Delta\dot{X}} + \underset{(2mn,\ 3n)}{\ddot{A}} \ \underset{(3n,\ 1)}{\Delta\ddot{X}} - \underset{(2mn,\ 1)}{b} = \underset{(2mn,\ 1)}{v}$$

Adding Observation of Exterior Orientation Parameters

$$\begin{bmatrix} X^c \\ Y^c \\ Z^c \\ \omega^c \\ \varphi^c \\ \kappa^c \end{bmatrix}_i = \begin{bmatrix} X^c \\ Y^c \\ Z^c \\ \omega^c \\ \varphi^c \\ \kappa^c \end{bmatrix}_i^{oo} + \begin{bmatrix} v_{X^c} \\ v_{Y^c} \\ v_{Z^c} \\ v_\omega \\ v_\varphi \\ v_\kappa \end{bmatrix}_i$$

Observations of camera parameters

$$\begin{bmatrix} \hat{X}^c \\ \hat{Y}^c \\ \hat{Z}^c \\ \hat{\omega} \\ \hat{\phi} \\ \hat{\kappa} \end{bmatrix}_i = \begin{bmatrix} X^c \\ Y^c \\ Z^c \\ \omega^c \\ \varphi^c \\ \kappa^c \end{bmatrix}_i^{o} + \begin{bmatrix} \Delta \hat{X}^c \\ \Delta \hat{Y}^c \\ \Delta \hat{Z}^c \\ \Delta \hat{\omega}^c \\ \Delta \hat{\varphi}^c \\ \Delta \hat{\kappa}^c \end{bmatrix}_i$$

Parameters in terms of approximate values and corrections

RHS of these 2 equations are equated

Hence

$$\begin{bmatrix} \Delta\hat{X}^c \\ \Delta\hat{Y}^c \\ \Delta\hat{Z}^c \\ \Delta\hat{\omega}^c \\ \Delta\hat{\varphi}^c \\ \Delta\hat{\kappa}^c \end{bmatrix}_i - \left\{ \begin{bmatrix} X^c \\ Y^c \\ Z^c \\ \omega^c \\ \varphi^c \\ \kappa^c \end{bmatrix}_i^{oo} - \begin{bmatrix} X^c \\ Y^c \\ Z^c \\ \omega^c \\ \varphi^c \\ \kappa^c \end{bmatrix}_i^{o} \right\} = \begin{bmatrix} v_X^c \\ v_Y^c \\ v_Z^c \\ v_\omega^c \\ v_\varphi^c \\ v_\kappa^c \end{bmatrix}_i$$

$$\Delta \dot{X} \quad - \quad b_2 = \quad v_2$$

Equations for observations of control point coordinates are determined in a similar manner

$$\begin{pmatrix} \Delta\hat{X} \\ \Delta\hat{Y} \\ \Delta\hat{Z} \end{pmatrix}_j - \begin{pmatrix} X^{oo} - X^o \\ Y^{oo} - Y^o \\ Z^{oo} - Z^o \end{pmatrix}_j = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_j$$

$$\ddot{\Delta} X \quad - \quad b_3 = \quad v_3$$

The three sets of equations are compiled:

1. linearised form of the collinearity equations for all observed points
2. observation equations for the camera parameters
3. observation equations for the control points

$$\dot{A}\,\Delta\dot{X} + \ddot{A}\,\Delta\ddot{X} - b_1 = v$$ Collinearity

$$\Delta\dot{X} - b_2 = v_2$$ Observations of Camera parameters

$$\Delta\ddot{X} - b_3 = v_3$$ Observations of control points

Example of observation equations

Adding Covariance Matrices of the Observations

The quality of the observations are determined by their variances (standard deviation)^2

Variances must be determined for all observations for:

1. Image coordinate observations

2. Camera parameter observations

3. Control point observations

The assumption with all observations is that there is no correlation between any of the observations

Variances of coordinate observations of point j on photograph 1

$$Q_{1j} = \begin{bmatrix} \sigma^2_{x_j} & 0 \\ 0 & \sigma^2_{y_j} \end{bmatrix}$$

Hence

$$P_{1j} = Q_{1j}^{-1}$$

P-matrix for all coordinate observations

$$\underset{2mn,2mn}{\mathbf{P}_1} = \begin{bmatrix} \mathbf{P}_{11} & & & & \\ & \mathbf{P}_{12} & & & \\ & & . & & \\ & & & . & \\ & & & & \mathbf{P}_{1n} \end{bmatrix}$$

The P-matrix of the camera parameters is derived from the inverse of variance/covariance matrix

$$P2i = \begin{bmatrix} \sigma^2_{X^c} & & & & & \\ & \sigma^2_{Y^c} & & & & \\ & & \sigma^2_{Z^c} & & & \\ & & & \sigma^2_{\omega} & & \\ & & & & \sigma^2_{\varphi} & \\ & & & & & \sigma^2_{\kappa} \end{bmatrix}^{-1}$$

No correlation exists between each of the exterior orientation parameters

Therefore the P matrix for camera parameter observations

$$\underset{6m,6m}{\mathbf{P}_2} = \begin{bmatrix} \mathbf{P}_{21} & & & & & \\ & \mathbf{P}_{22} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \mathbf{P}_{2m-1} & \\ & & & & & \mathbf{P}_{2m} \end{bmatrix}$$

Camera parameter observations

The inverse of covariance matrix of the observations of the control point coordinates of point j can be written as

$$P_{3j} = \begin{bmatrix} \sigma_{X_j}^2 & & \\ & \sigma_{Y_j}^2 & \\ & & \sigma_{Z_j}^2 \end{bmatrix}^{-1}$$

No correlation exists between each of the control point parameters

Hence, the P Matrix for the control point observations is:

$$\underset{3\mathbf{n'},3\mathbf{n'}}{\mathbf{P}_3} = \begin{bmatrix} \mathbf{P}_{31} & & & & & \\ & \mathbf{P}_{32} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \mathbf{P}_{3\mathbf{n'}-1} & \\ & & & & & \mathbf{P}_{3\mathbf{n'}} \end{bmatrix}$$

The set of equations with P-matrices added

$$
\begin{bmatrix} \dot{A} & \ddot{A} \\ I & 0 \\ 0 & I \end{bmatrix}
\begin{bmatrix} \Delta \dot{X} \\ \Delta \ddot{X} \end{bmatrix}
-
\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
=
\begin{bmatrix} v \\ v_2 \\ v_3 \end{bmatrix}
\quad ; \quad
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}
$$

$$
A \quad \Delta X \quad - \quad b \quad = \quad v \quad ; \quad P
$$

Normal equations for the least squares adjustment

$$
A^T P A \; \Delta X = A^T P b
$$

Expanding the normal equations with each of the sub-matrices

$$\begin{bmatrix} \dot{A}^T P_1 \dot{A} + P_2 & \dot{A}^T P_1 \ddot{A} \\ \ddot{A}^T P_1 \dot{A} & \ddot{A}^T P_1 \ddot{A} + P_3 \end{bmatrix} \begin{bmatrix} \Delta \dot{X} \\ \Delta \ddot{X} \end{bmatrix} = \begin{bmatrix} \dot{A}^T P_1 b_1 + P_2 b_2 \\ \ddot{A}^T P_1 b_1 + P_3 b_3 \end{bmatrix}$$

Written simply $\quad N \ \Delta X = \quad T$

- The solution of these equations can result in very large matrices with many zeros
- Solution is divided into 2 steps where the vector $\dot{\Delta}$ is solved first
- Then $\ddot{\Delta}$ will be solved
- This means that a set of Reduced Normal Equations in terms of only $\dot{\Delta}$ is derived from the above:

Let the matrix be expressed in terms of the sub-matrices as follows:

$$\begin{bmatrix} N_{11} & N_{12} \\ N_{12}^T & N_{22} \end{bmatrix} \begin{bmatrix} \dot{\Delta} \\ \ddot{\Delta} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

Forming the matrices into 2 equations comprising the sub-matrices

$$N_{11}\dot{\Delta} + N_{12}\ddot{\Delta} = T_1$$

$$N_{12}^T\dot{\Delta} + N_{22}\ddot{\Delta} = T_2$$

Pre-multiply the 2nd equation by N_{22}^{-1}

$$N_{22}^{-1} \times \; N_{12}^{T}\dot{\Delta} + N_{22}\ddot{\Delta} = T_{2}$$

$$N_{22}^{-1}N_{12}^{T}\dot{\Delta} + N_{22}^{-1}N_{22}\ddot{\Delta} = N_{22}^{-1}T_{2}$$

$$N_{22}^{-1}N_{12}^{T}\dot{\Delta} + \ddot{\Delta} = N_{22}^{-1}T_{2}$$

Therefore $\ddot{\Delta} = N_{22}^{-1}T_{2} - N_{22}^{-1}N_{12}^{T}\dot{\Delta}$

Substituting the above formula into the first of the 2 equations will give the equation for $\dot{\Delta}$

Reduced Normal Equations

$$(N_{11} - N_{12} N_{22}^{-1} N_{12}^{T})\dot{\Delta} = T_1 - N_{12} N_{22}^{-1} T_2$$

- The Reduced Normal Equations have a special characteristic
- They can be solved more efficiently no matter how many unknowns have to be solved.
- A typical pattern of the Reduced Normal Equations
- Bandwidth minimization can be undertaken to reduce the size of the matrix
- Achieved by strategic number the photos in the adjustment
- Inverse of the Normal Equation Matrix will give the estimated accuracy of computed parameters
- Estimates of accuracies of observations can also be determined

1	2	3	4	5	6	7	8	9	10	11	12
+	+	+	+	+	+	+	+	+	+	+	+
13	14	15	16	17	18	19	20	21	22	23	24
+	+	+	+	+	+	+	+	+	+	+	+
25	26	27	28	29	30	31	32	33	34	35	36
+	+	+	+	+	+	+	+	+	+	+	+

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
1	×	×	×										×	×	×																					
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Space Intersection

- Once the values of camera parameters
- Object coordinates of any additional points in two or more images can be computed from the collinearity equations of these images.
- This is referred to as Space Intersection
- In this case, only X_j, Y_j and Z_j are the unknowns.
- There are 4 equations with 3 unknowns
- Least squares adjustment is required

The solution involves an equation of the form

$$A \, \Delta X \quad - \; b \quad = v$$

A = coefficient matrix, linearized in terms of Xj, Yj and Zj

ΔX = corrections to unknowns Xj, Yj and Zj

v = residual vector

$$A = \frac{-f}{W} \begin{bmatrix} 1 & 0 & -\frac{U}{W} \\ 0 & 1 & -\frac{V}{W} \end{bmatrix} M$$

Expanding out this matrix multiplication gives

$$\mathbf{A} = \frac{-f}{W}\begin{bmatrix} m_{11} - \frac{U}{W}m_{31} & m_{12} - \frac{U}{W}m_{32} & m_{13} - \frac{U}{W}m_{33} \\ m_{21} - \frac{V}{W}m_{31} & m_{22} - \frac{V}{W}m_{32} & m_{23} - \frac{V}{W}m_{33} \end{bmatrix}$$

The equations for the solution of the intersection are:

$$\frac{-f}{W}\begin{bmatrix} m_{11} - \frac{U}{W}m_{31} & m_{12} - \frac{U}{W}m_{32} & m_{13} - \frac{U}{W}m_{33} \\ m_{21} - \frac{V}{W}m_{31} & m_{22} - \frac{V}{W}m_{32} & m_{23} - \frac{V}{W}m_{33} \end{bmatrix}\begin{bmatrix} \Delta X_j \\ \Delta Y_j \\ \Delta Z_j \end{bmatrix} - \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Where

$$U = m_{11}(X_j - X^c) + m_{12}(Y_j - Y^c) + m_{13}(Z_j - Z^c)$$

$$V = m_{21}(X_j - X^c) + m_{22}(Y_j - Y^c) + m_{23}(Z_j - Z^c)$$

$$W = m_{31}(X_j - X^c) + m_{32}(Y_j - Y^c) + m_{33}(Z_j - Z^c)$$

Equations for the left hand photo

$$\frac{-f}{W}\left(m_{11}-\frac{U}{W}m_{31}\right)\Delta X_J \quad \frac{-f}{W}\left(m_{12}-\frac{U}{W}m_{32}\right)\Delta Y_J \quad \frac{-f}{W}\left(m_{13}-\frac{U}{W}m_{33}\right)\Delta Z_J \quad -(x_j - A_x^0) = v_x$$

$$\frac{-f}{W}\left(m_{21}-\frac{U}{W}m_{31}\right)\Delta X_J \quad \frac{-f}{W}\left(m_{22}-\frac{U}{W}m_{32}\right)\Delta Y_J \quad \frac{-f}{W}\left(m_{23}-\frac{U}{W}m_{33}\right)\Delta Z_J \quad -(y_j - A_y^0) = v_y$$

For the right hand photo

$$\frac{-f}{W}\left(m_{11}-\frac{U}{W}m_{31}\right)\Delta X_J \quad \frac{-f}{W}\left(m_{12}-\frac{U}{W}m_{32}\right)\Delta Y_J \quad \frac{-f}{W}\left(m_{13}-\frac{U}{W}m_{33}\right)\Delta Z_J \quad -(x_j - A_x^0) = v_x$$

$$\frac{-f}{W}\left(m_{21}-\frac{U}{W}m_{31}\right)\Delta X_J \quad \frac{-f}{W}\left(m_{22}-\frac{U}{W}m_{32}\right)\Delta Y_J \quad \frac{-f}{W}\left(m_{23}-\frac{U}{W}m_{33}\right)\Delta Z_J \quad -(y_j - A_y^0) = v_y$$

Where

$$\begin{bmatrix} A_x \\ A_y \end{bmatrix} = \begin{bmatrix} m_{11}-\frac{U}{W}m_{31} & m_{12}-\frac{U}{W}m_{32} & m_{13}-\frac{U}{W}m_{33} \\ m_{21}-\frac{V}{W}m_{31} & m_{22}-\frac{V}{W}m_{32} & m_{23}-\frac{V}{W}m_{33} \end{bmatrix} \begin{bmatrix} X_j^0 \\ Y_j^0 \\ Z_j^0 \end{bmatrix}$$

And ^{0} refers to the approximate values of X_j, Y_j, Z_j

Intersection solution

P_1 = Inverse of the covariance matrix of image coordinates previously derived in equation 3.38

$$\Delta = (A^T P_1 A)^{-1} (A^T P_1 b)$$

Δ comprises 3 unknowns which must be solved by iteration

Error detection and elimination

- Errors in observations may be a combination of:
- Accidental
  - assumed to normally follow a normal distribution and are handled by the least squares solution
- Systematic
  - aim to models these errors so that they are largely eliminated before the least squares adjustment is undertaken,
  - residual systematic errors are modelled during the adjustment by self-calibration
- Blunders
  - mistakes and may be large
  - will be revealed during the least squares adjustment
  - must be eliminated to achieve a solution – data snooping

Self Calibration in Block Adjustment

- Self calibration is designed to correct for unknown systematic errors in the images.
- Modelled by appropriate parameters in the terms Δx_{ij} and Δy_{ij} in the collinearity equations
- The formulations are based on assumptions as to their form, but they are really unknown.
- Several sets of equations have been suggested to describe these potential systematic errors
- Different for film and digital images
- Brown (1976) parameters are used in software to solve ADS80 (ADS40) images using ORIMA, but adapted to digital images

Known Systematic Errors in Aerial Imaging

- Radial lens distortion - usually known from calibration as shown on in Chapter 1.
  - A typical formula is:

$$dr = K_1 r^3 + K_2 r^5 + K_3 r^7 + \ldots$$

$$\text{where } r^2 = (x - x_o)^2 + (y - y_o)^2$$

- Decentring lens distortion caused by inexact alignment of lens components – usually negligible in aerial images
- For digital images, there may be errors in the pixel mosaic
- Earth curvature – derived from known equations
- Refraction – estimated from standard atmosphere

Gruen's parameter set for digital images

$$\Delta x_{ij} = -\Delta x_0 + \Delta c.(x-x_0)/c + (x-x_0)Sx + (y-y_0).a + (x-x_0).r^2.k_1 + (x-x_0).r^4k_2 + (x-x_0).r^6.k_3 + \{r^2 + 2.(x-x_0)^2\}.P_1 + 2(x-x_0).(y-y_0).P_2$$

$$\Delta y_{ij} = -\Delta y_0 + \Delta c.(y-y_0)/c + 0 + (x-x_0).a + (y-y_0).r^2.k_1 + (y-y_0).r^4k_2 + (y-y_0).r^6.k_3 + 2(x-x_0).(y-y_0).P_1 + \{r^2 + 2.(y-y_0)^2\}.P_2$$

Where $r^2 = (x-x_0)^2 + (y-y_0)^2$

Other equations

- Fritsch:

Fourier series with 16 parameters which are said to be theoretically preferred

R. Tang, D. Fritsch, M. Cramer New rigorous and flexible Fourier self-calibration models for airborne camera calibration, ISPRS Journal of Photogrammetry and Remote Sensing 71 (2012): 76–85

Blunder detection – Data Snooping

- Since observations are often highly correlated, the impact of an error at a point will distributed to other points
- Multiple blunders are often difficult to isolate
- It may be necessary to do several runs of an adjustment to locate all blunders
- Some solutions are available for dealing with multiple errors

Blunder detection – robust estimation

- A method that aims to automatically eliminate the blunders
- Applies weights to observation according to the magnitude of the residuals after each iteration in the adjustment
- Weights are inversely proportional to magnitude of residuals
  - Observations with near zero residuals should receive a large weight
  - Observations with large residuals should receive low weights
- Typical weighting function p:

$$p = \frac{1}{1 + (a.|v|)^{b}}$$

- Where v is the observation residual after an iteration
-    and a>0 and b > 0
- Block adjustment software may include robust estimation as an option

Exterior Orientation of Aerial Pushbroom and Whiskbroom Images

1. Simple transformations for 2D only

Uses an empirical function based on a minimum of GCPs for a 2D transformation only

$$X_T = a_1 + a_2.x + a_3.y + a_4 x^2 + a_5 x^3 + a_6 x.y + a_7 x^2.y + a_8 x.y^2 + a_9 y^2 + a_{10} y^3$$

$$Y_T = b_1 + b_2.x + b_3.y + b_4 x^2 + b_5 x^3 + b_6 x.y + b_7 x^2.y + b_8 x.y^2 + b_9 y^2 + b_{10} y^3$$

where X_T, Y_T are the transformed object coordinates

x and y are the image coordinates

$a_1, a_1 \ldots\ldots b_{10}$ are unknown parameters of the transformation

These transformations are usually more suited to satellite images since distortions due to elevations are likely to be smaller. Sub-pixel accuracy usually achievable.

1. Simple transformations for 2D only

These formulas incorporating effects of elevations are:

DLT (well-known from close range photogrammetry)

$$x = (L_1X + L_2Y + L_3Z + L4)/(L_9X + L_{10}Y + L_{11}Z + 1)$$

$$y = (L_5X + L_6Y + L_7Z + L_8)/(L_9X + L_{10}Y + L_{11}Z + 1)$$

Affine transformation with elevation correction.

$$x = A_1X + A_2Y + A_3Z + A_4$$

$$y = A_5X + A_6Y + A_7Z + A_8$$

2. Rigorous Solution for PushBroom and Whiskbroom Scanners

Collinearity equations can be based on the assumption that each individual scan line in the image is recorded instantaneously.

A set of collinearity equations is defined for each scan line.

Assumptions made about the changes in exterior parameters or the are based on GNSS/IMU data

[Adjustment of ADS80/100 pushbroom images]

[Adjustment of satellite images]